Understanding Generalized Hopf Bandings through Hopf Plumbings

CSUF | COLLEGE OF Natural Sciences and Mathematics

Jordan Incledon and Dr. Matt Rathbun (Advisor)
California State University, Fullerton

Introduction

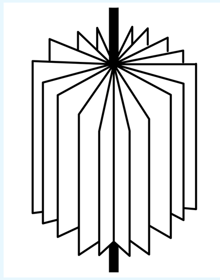
Knot Theory studies mathematical properties of knots. We investigate properties of two special operations on rare types of surfaces that have knots as their boundary.

Knots and Seifert Surfaces

A **knot** is a simple closed curve in S^3. A **link** is a disjoint union of knots. A **Seifert surface** of a link L is an oriented surface embedded in S^3 with boundary L.

Fibration and Monodromy

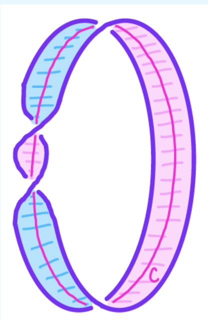
A **fiber surface** F is a rare type of Seifert surface. We can think of the **fibration** by F as $F \times [0, 1]$ with $(F, 0)$ and $(F, 1)$ identified by a homeomorphism called the **monodromy**.



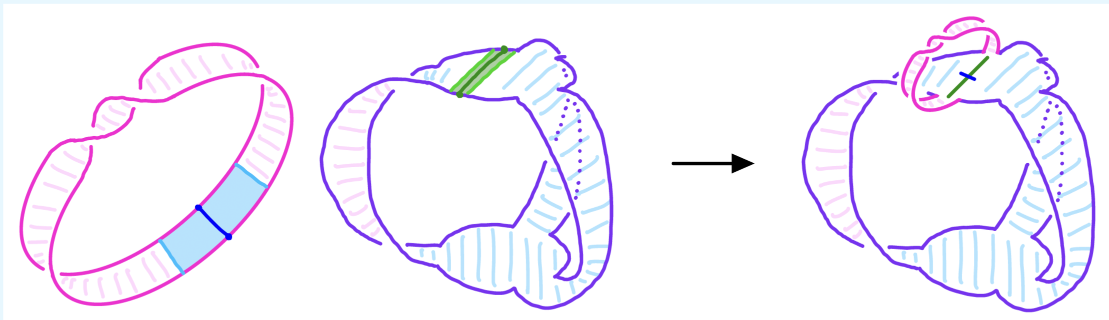
A local picture of fibration. The "sheets" extend in all directions.

Hopf Plumbings and Generalized Hopf Bandings

A **Hopf band** is an annulus with a full twist and is a fiber surface. In a **Hopf plumbing**, we attach a Hopf band in a particular way to a fiber surface. Its inverse is called a **de-plumbing.**



Hopf band with core curve c

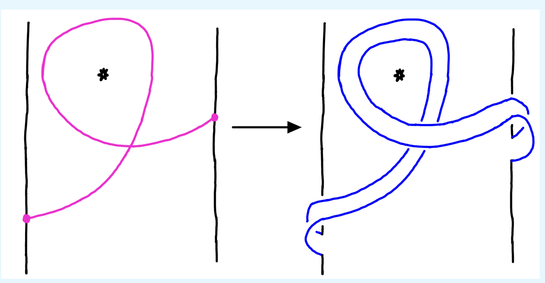


Hopf plumbing on the trefoil

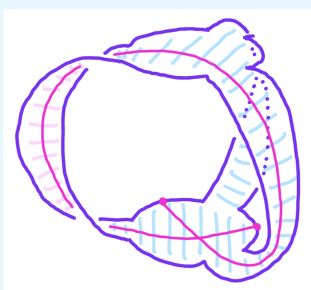
Theorem (Giroux) [1]

Any two fiber surfaces are related via a sequence of Hopf plumbings and then de-plumbings.

In a **generalized Hopf banding**, we attach a band parallel to a surface so the band crosses itself once.



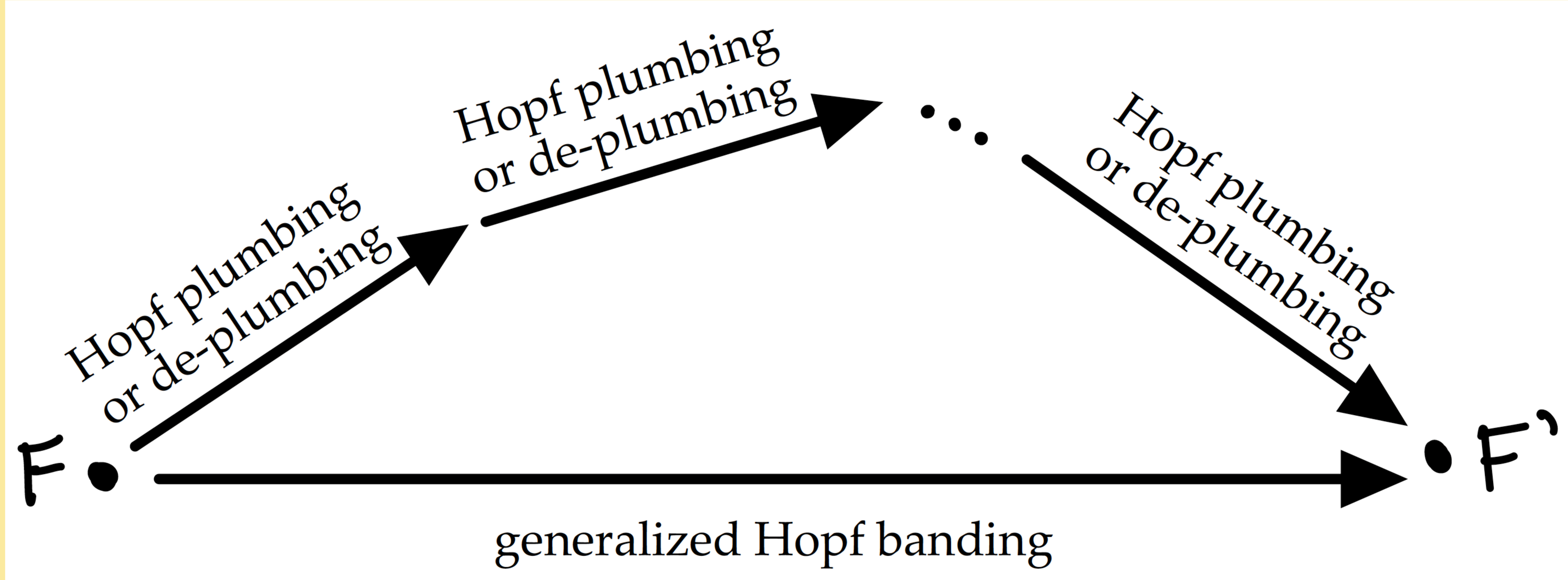
An abstraction of the generalized Hopf banding



We could do a generalized Hopf banding along the pink arc

Question

How can we represent a generalized Hopf banding as a sequence of Hopf plumbings and de-plumbings?



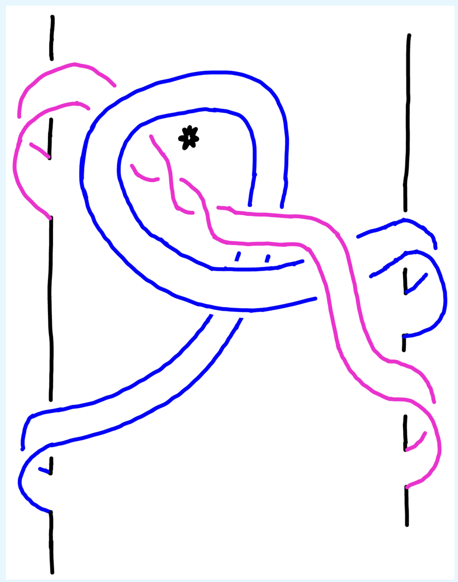
Results

We have proven a result that will inform further explorations:

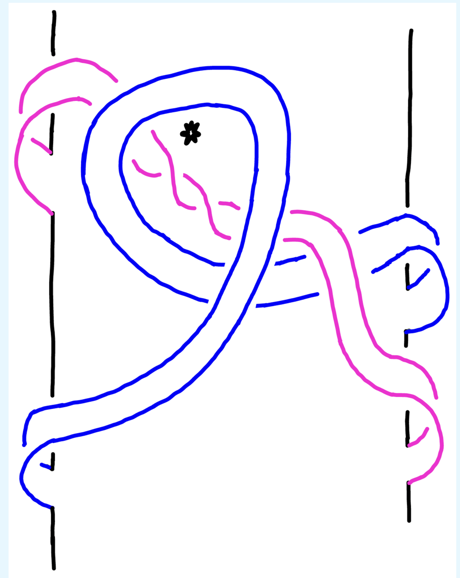
Theorem

Performing a pair of *specific* Hopf plumbings results in the same surface as performing a pair of *specific* generalized Hopf bandings.

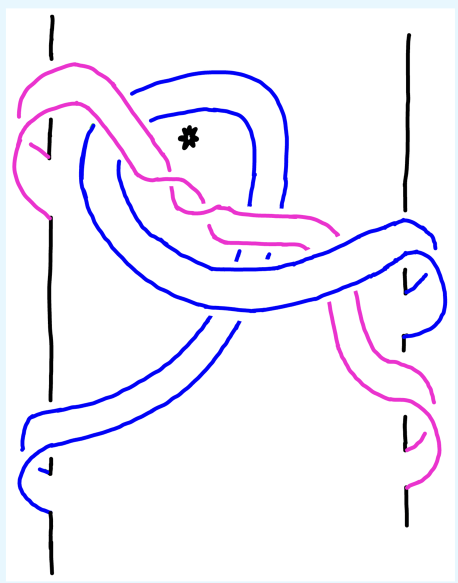
We have identified eight such surfaces:



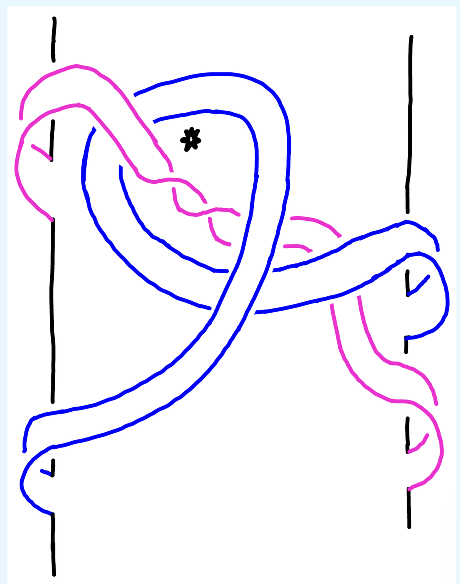
Surface A



Surface B



Surface C

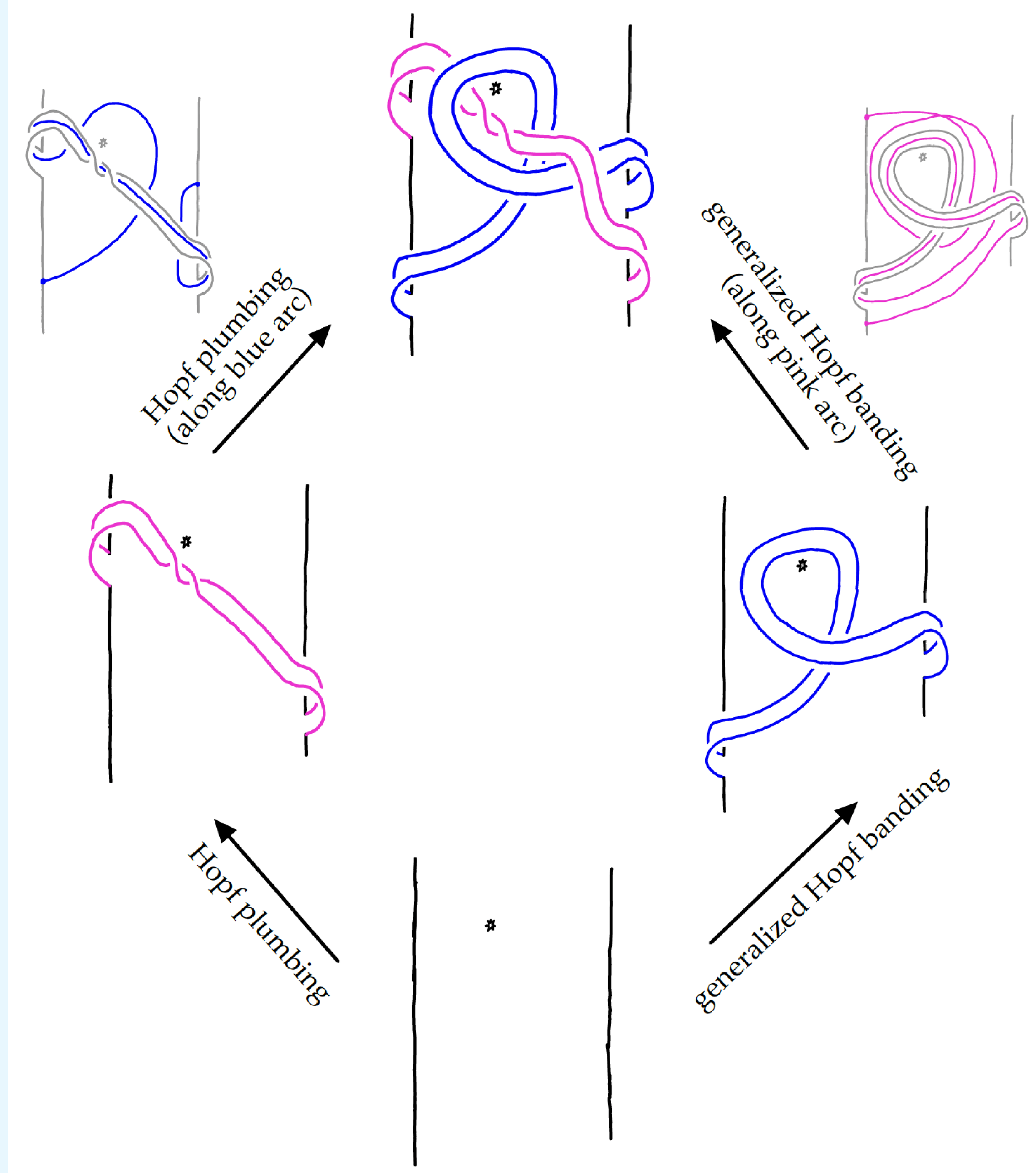


Surface D

*The star represents potential topological complexity.

The remaining four are given by symmetry along a vertical axis of the above four.

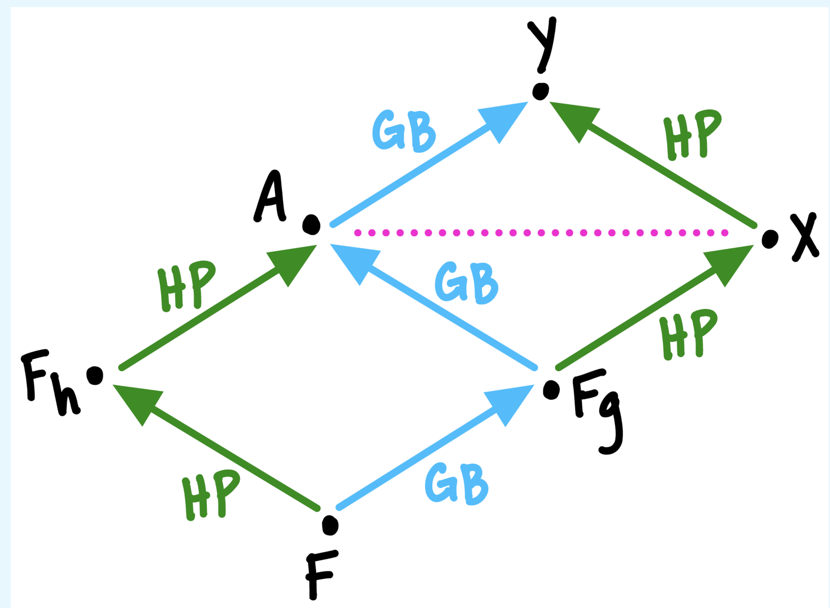
Construction of Surface A



Construction of Surface A

Extending the Relationship

We build upon the relationship introduced above to create more potential pathways between relevant surfaces.



"HP" indicates a Hopf plumbing.

"GB" indicates a generalized Hopf banding.

Acknowledgements and References

Thank you to all those who generously provided funding for this project—CSUF Math Department MSRP, LSAMP (NSF grant #1826490), Building Bridges to the Professoriate (a UCR-Cal State collaboration grant funded by the UC-HSI Doctoral Diversity Initiative), and Dr. Paul and Sara DeLand.

[1] Giroux, E. and Goodman, N. (2006). On the stable equivalence of open books in three-manifolds. *Geometry and Topology 10, 97-114.*